Distance in the Affine Buildings of SL_n and Sp_n

Alison Setyadi
setyadi@member.ams.org

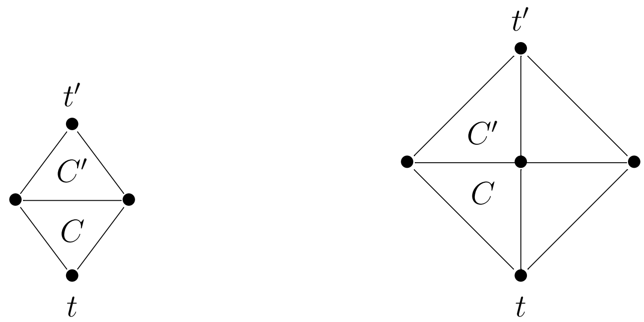
Abstract

For a local field K and $n \geq 2$, let Ξ_n and Δ_n denote the affine buildings naturally associated to the special linear and symplectic groups $\mathrm{SL}_n(K)$ and $\mathrm{Sp}_n(K)$, respectively. We relate the number of vertices in Ξ_n ($n \geq 3$) close (i.e., gallery distance 1) to a given vertex in Ξ_n to the number of chambers in Ξ_n containing the given vertex, proving a conjecture of Schwartz and Shemanske. We then consider the special vertices in Δ_n ($n \geq 2$) close to a given special vertex in Δ_n (all the vertices in Ξ_n are special) and establish analogues of our results for Δ_n.

Introduction

A building is a finite-dimensional simplicial complex in which any two of its chambers (maximal simplices) can be connected by a gallery. In other words, if Δ is a building, then for any chambers $C, D \in \Delta$, there is a sequence $C = C_0, C_1, \ldots, C_m = D$ of chambers in Δ such that C_i and C_{i+1} are adjacent (share a codimension-one face) for all $0 \leq i \leq m-1$; in this case, the number m is the length of the gallery $C_0, \ldots, C_m$. The combinatorial distance between C and D is the minimal length of a gallery in Δ connecting C and D (see [1, p. 14]). Following [1, p. 15], define the *distance* between any non-empty simplices $A, B \in \Delta$ to be the minimal length of a gallery in Δ starting at a chamber containing A and ending at a chamber containing B (cf. [6, p. 125]). Then the vertices $t, t' \in \Delta$ are distance one apart or *close* if and only if there are adjacent chambers $C, C' \in \Delta$ such that $t \in C$, $t' \in C'$, but $t, t' \notin C \cap C'$ (the simplex shared by C and C'); i.e., if and only if t and t' are in adjacent chambers in Δ but not a common one (cf. [6, p. 127]). Figures 1(a) and 1(b) show close vertices in the affine buildings naturally associated to $\mathrm{SL}_3(K)$ and $\mathrm{Sp}_2(K)$, respectively, for any local field K. Note that if Δ is a building and $t, t' \in \Delta$ are close vertices, then as vertices in the underlying graph of Δ, t and t' are not graph distance 1 apart but are always graph distance 2 apart.

Let K be a local field with valuation ring $\mathcal{O}$, uniformizer π, and residue field $k \cong \mathbb{F}_q$, and let Ξ_n denote the affine building naturally associated to $\mathrm{SL}_n(K)$. In [6, Theorem 3.3], Schwartz and Shemanske show that for all $n \geq 3$, the number ω_n of vertices in Ξ_n close to a given vertex in Ξ_n is the number of right cosets of $\mathrm{GL}_n(\mathcal{O})$ in $\mathrm{GL}_n(\mathcal{O})\mathrm{diag}(1, \pi, \ldots, \pi, \pi^2)\mathrm{GL}_n(\mathcal{O})$; i.e., the Hecke operator $\mathrm{GL}_n(\mathcal{O})\mathrm{diag}(1, \pi, \ldots, \pi, \pi^2)\mathrm{GL}_n(\mathcal{O})$ acts as a generalized adjacency operator on Ξ_n. They also conjecture that for all $n \geq 3$, $q \cdot r_n = r_{n-2}\, \omega_n$, where r_n is the number of chambers in Ξ_n containing a given vertex, with $r_1 := 1$ (see the remark following [6, Proposition 3.4]).



(a) Two close vertices in Ξ_3. (b) Two close vertices in Δ_2.

Figure 1: Examples of close vertices.

In Section 1, we prove Schwartz and Shemanske's conjecture in two ways. Our first approach is via module theory. More precisely, we use the description of the chambers in Ξ_n in terms of lattices in an n-dimensional K-vector space (see, for example, [5, p. 115]) to obtain an explicit formula for ω_n (Proposition 1.1); together with Schwartz and Shemanske's formula for r_n [6, Proposition 2.4], this proves Theorem 1.1. Our second approach is through combinatorics (Theorem 1.2). Specifically, we show that if $t, t' \in \Xi_n$ are close vertices, then there is a one-to-one correspondence between the galleries of length 1 in Ξ_n whose initial chamber contains t and whose ending chamber contains t' and the chambers in the spherical $A_{n-3}(k)$ building. This gives an explanation for the relationship between ω_n and r_n in terms of the structure of Ξ_n. In Section 2, we consider the special vertices in the affine building Δ_n naturally associated to $\mathrm{Sp}_n(K)$ ($n \geq 2$) close to a given special vertex in Δ_n (all the vertices in Ξ_n are special). Using the fact that Δ_n is a subcomplex of Ξ_{2n}, we adapt the proofs of the results for close vertices in Ξ_{2n} to prove analogues for Δ_n. In particular, we establish analogues of [6, Theorem 3.3] and Theorem 1.1 (Theorems 2.1 and 2.2, respectively) and a partial analogue of Theorem 1.2 (Proposition 2.11). Note that while every vertex in Ξ_{2n} is special, only two vertices in each chamber in Δ_n are special; hence, our analysis for Δ_n requires more care than that needed for Ξ_{2n}.

After proving Theorems 1.1 and 2.2, we learned that the formulas in Propositions 1.1 and 2.9 are both special cases of a result of Parkinson [4, Theorem 5.15] and that the formula in Proposition 1.1 also follows from a result of Cartwright [2, Lemma 2.2]. We view the buildings Ξ_n and Δ_n as combinatorial objects naturally associated to $\mathrm{SL}_n(K)$ and $\mathrm{Sp}_n(K)$, respectively, and make use of the lattice descriptions of these buildings (see [3] and [5]). As a result, our methods require little more than the definition of a building—namely, some module theory. In contrast to our approach, Cartwright views Ξ_n in terms of hyperplanes, affine transformations, and convex hulls, and Parkinson considers buildings via root systems and Poincaré polynomials of Weyl groups. The numbers ω_n and $\omega(\Delta_n)$ that we use are special cases of Parkinson's N_λ, which he uses to define vertex set averaging operators on arbitrary locally finite, regular affine buildings and whose formula he uses to prove results about those operators.

I thank Paul Garrett for the idea behind the proof of Proposition 1.1, and hence that of Proposition 2.9. Finally, the results contained here form part of my doctoral thesis, which I wrote under the guidance of Thomas R. Shemanske.

1 Close Vertices in the Affine Building Ξ_n of $\mathrm{SL}_n(K)$

From now on, K is a local field with discrete valuation "ord," valuation ring $\mathcal{O}$, uniformizer π, and residue field $k \cong \mathbb{F}_q$. For any finite-dimensional K-vector space V, define a *lattice* in V to be a free $\mathcal{O}$-submodule of V of rank $\dim_K V$, with two lattices L and L' in V *homothetic* if $L' = \alpha L$ for some $\alpha \in K^\times$; write $[L]$ for the homothety class of the lattice L.

The affine building Ξ_n naturally associated to $\mathrm{SL}_n(K)$ can be modeled as an $(n-1)$-dimensional simplicial complex as follows (see [5, p. 115]). Let V be an n-dimensional K-vector space. Then a *vertex* in Ξ_n is a homothety class of lattices in V, and two vertices $t, t' \in \Xi_n$ are *incident* if there are representatives $L \in t$ and $L' \in t'$ such that $\pi L \subseteq L' \subseteq L$; i.e., such that $L'/\pi L$ is a k-subspace of $L/\pi L$. Thus, a *chamber* (maximal simplex) in Ξ_n has n vertices $t_0, \ldots, t_{n-1}$ with representatives $L_i \in t_i$ such that $\pi L_0 \subsetneq L_1 \subsetneq \cdots \subsetneq L_{n-1} \subsetneq L_0$ and $[L_1 : \pi L_0] = q = [L_i : L_{i-1}]$ for all $2 \le i \le n-1$. From now on, write that a chamber in Ξ_n corresponds to the chain $\pi L_0 \subsetneq L_1 \subsetneq \cdots \subsetneq L_{n-1} \subsetneq L_0$ only when the lattices $L_0, \ldots, L_{n-1}$ satisfy the conditions in the last sentence.

For the rest of this section, $n \ge 3$. Let $t \in \Xi_n$ be a vertex with representative L. Then a chamber $C \in \Xi_n$ containing t corresponds to a chain of the form

$$\pi L \overset{q}{\subsetneq} L_1 \overset{q}{\subsetneq} \cdots \overset{q}{\subsetneq} L_{n-1} \overset{q}{\subsetneq} L \tag{1}$$

(cf. [3, p. 323]). The codimension-one face in C not containing t thus corresponds to the chain

$$L_1 \overset{q}{\subsetneq} \cdots \overset{q}{\subsetneq} L_{n-1},$$

and a vertex in Ξ_n is close to t if it has a representative $M \neq L$ such that

$$\pi M \overset{q}{\subsetneq} L_1 \overset{q}{\subsetneq} \cdots \overset{q}{\subsetneq} L_{n-1} \overset{q}{\subsetneq} M. \tag{2}$$

Given the lattices L_1 and L_{n-1}, the possible L and M satisfy $L_{n-1} \subsetneq L \neq M \subsetneq \pi^{-1} L_1$. On the other hand, if $t, t' \in \Xi_n$ are close vertices, then there must be representatives $L \in t$ and $M \in t'$ and lattices $L_1, \ldots, L_{n-1}$ as in (1) such that $L_{n-1} \subsetneq L \neq M \subsetneq \pi^{-1} L_1$. Recall that if M_1 and M_2 are free, rank n, $\mathcal{O}$-modules with $M_1 \subseteq M_2$, then $M_1 \subseteq M' \subseteq M_2$ implies M' is also a free, rank n, $\mathcal{O}$-module. Thus, both $L \cap M$ and $L + M$ are lattices in V. Furthermore, $L \neq M$ and $[L : L_{n-1}] = q = [M : L_{n-1}]$ imply $L \cap M = L_{n-1}$ and $L + M = \pi^{-1} L_1$, but we can vary $L_2, \ldots, L_{n-2}$ as long as $L_1 \subsetneq L_2 \subsetneq \cdots \subsetneq L_{n-2} \subsetneq L_{n-1}$. In other words, if t and t' are close vertices in Ξ_n, there may be two (or more) pairs of adjacent chambers C and C' in Ξ_n with $t \in C$, $t' \in C'$, but $t, t' \notin C \cap C'$ (see Figure 2). We return to this later.

Before we count the number of vertices in Ξ_n close to a given vertex $t \in \Xi_n$, we make a few observations. Fix a representative $L \in t$. Since $L/\pi L \cong k^n$, the Correspondence Theorem and the fact that any $\mathcal{O}$-submodule of L containing πL is a lattice in V imply that the number of L_1 is the number of 1-dimensional k-subspaces of $L/\pi L$. Similarly, given L_1 as above, the number of lattices L_{n-1} with $L_1 \subsetneq L_{n-1} \subsetneq L$ and $[L : L_{n-1}] = q$ is the number of $(n-2)$-dimensional k-subspaces of $L/L_1 \cong k^{n-1}$. Finally, given L_1 and L_{n-1} as above, the number of lattices $M \neq L$ such that $L_{n-1} \subsetneq M \subsetneq \pi^{-1} L_1$ is one less than the number of non-trivial, proper k-subspaces of $\pi^{-1} L_1 / L_{n-1} \cong k^2$.

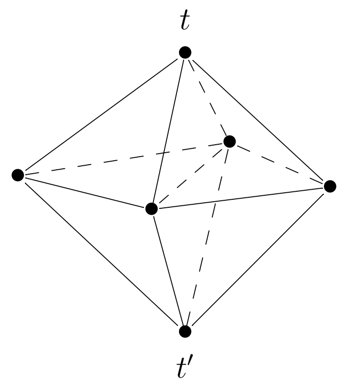


Figure 2: Two close vertices in Ξ_4.

Proposition 1.1. *If $t \in \Xi_n$ is a vertex, then the number ω_n of vertices in Ξ_n close to t is*

$$\frac{q^n - 1}{q - 1} \cdot \frac{q^{n-1} - 1}{q - 1} \cdot q$$

(independent of t).

Proof. This follows from the preceding comments, duality, and the fact that the number of 1-dimensional subspaces of $\mathbb{F}_q^m$ is exactly $(q^m - 1)/(q - 1)$. □

Corollary 1.1. *The number of right cosets of $\mathrm{GL}_n(\mathcal{O})$ in $\mathrm{GL}_n(\mathcal{O})\mathrm{diag}(1, \pi, \ldots, \pi, \pi^2)\mathrm{GL}_n(\mathcal{O})$ is $((q^n - 1)(q^{n-1} - 1) \cdot q)/(q - 1)^2$.*

Proof. This follows from [6, Theorem 3.3] and the last proposition. □

Let r_n be the number of chambers in Ξ_n containing a vertex $t \in \Xi_n$. Then [6, Proposition 2.4] and the last proposition establish the conjecture following Proposition 3.4 of [6]:

Theorem 1.1. *For all $n \geq 3$, $q \cdot r_n = r_{n-2}\ \omega_n$, where $r_1 = 1$.*

We now use the structure of Ξ_n to give a combinatorial proof for the relationship given in Theorem 1.1. Fix a vertex $t \in \Xi_n$. Then we can try to count the number of vertices in Ξ_n close to t by counting the number of galleries (in Ξ_n) of length 1 starting at a chamber containing t and ending at a chamber not containing t. By definition, there are r_n chambers $C \in \Xi_n$ containing t. Since a chamber in Ξ_n adjacent to C and not containing t must contain the codimension-one face in C not containing t, [3, p. 324] implies that there are q chambers in Ξ_n adjacent to C not containing t; hence, there are exactly $r_n \cdot q$ galleries of length 1 in Ξ_n whose initial chamber contains t and whose ending chamber does not contain t. On the other hand, if $t' \in \Xi_n$ is a vertex close to t, we count t' more than once if there is more than one gallery of length 1 in Ξ_n whose initial chamber contains t and whose ending chamber contains t' (see Figure 2); hence, $\omega_n = (r_n \cdot q)/m(t, t')$, where $m(t, t')$ is the number of galleries of length 1 in Ξ_n whose initial chamber contains t and whose ending chamber contains t'.

To determine $m(t, t')$, fix the following notation for the rest of this section. For close vertices $t, t' \in \Xi_n$, let $L \in t$, $M \in t'$ be representatives such that there are lattices $L_1, \ldots, L_{n-1}$

as in (1) and (2). Recall that $L_1 = \pi(L + M)$ and $L_{n-1} = L \cap M$, but we can vary $L_2, \ldots, L_{n-2}$ as long as $L_1 \subsetneq L_2 \subsetneq \cdots \subsetneq L_{n-2} \subsetneq L_{n-1}$. Since any gallery C, C' in Ξ_n such that $C = \{t, [L_1], \ldots, [L_{n-1}]\}$ and $C' = \{t', [L_1], \ldots, [L_{n-1}]\}$ satisfies $C \cap C' = \{[L_1], \ldots, [L_{n-1}]\}$, each gallery in Ξ_n counted by $m(t, t')$ is uniquely determined by the lattices $L_2, \ldots, L_{n-2}$. Define two vertices in Ξ_n to be *adjacent* if they are distinct and incident.

Proposition 1.2. *Let $t, t' \in \Xi_n$ be adjacent vertices. If $L \in t$, then there is a unique representative $L' \in t'$ such that $\pi L \subsetneq L' \subsetneq L$.*

Proof. Since t and t' are incident and $t \neq t'$, there are representatives $M \in t$ and $M' \in t'$ such that $\pi M \subsetneq M' \subsetneq M$. Moreover, M and L are homothetic, so $L = \alpha M$ for some $\alpha \in K^\times$; hence, $\pi L \subsetneq \alpha M' \subsetneq L$. Let $L' = \alpha M'$. If $L'' \in t'$ such that $\pi L \subsetneq L'' \subsetneq L$, let $\beta \in K^\times$ such that $L'' = \beta L'$. Suppose $\mathrm{ord}(\beta) = m$. Then $\pi L \subsetneq L' \subsetneq L$ implies $\pi^{m+1} L \subsetneq L'' \subsetneq \pi^m L$ and $L = \pi^m L$; i.e., $L'' = L'$. $\square$

Consider the set of vertices in Ξ_n that are adjacent to $t, t', [L+M]$, and $[L \cap M]$ (in the case $n = 3$, this set is empty), and define two such vertices to be incident if they are incident as vertices in Ξ_n. Let $\Xi_n^c(t, t')$ be the set consisting of

- the empty set,
- all vertices in Ξ_n adjacent to $t, t', [L+M]$, and $[L \cap M]$, and
- all finite sets A of vertices in Ξ_n adjacent to $t, t', [L+M]$, and $[L \cap M]$ such that any two vertices in A are adjacent.

Then $\Xi_n^c(t, t')$ is a simplicial complex. In particular, $\Xi_n^c(t, t')$ is a subcomplex of Ξ_n.

Lemma 1.1. *If $\emptyset \neq A \in \Xi_n^c(t, t')$ is an i-simplex, then A corresponds to a chain of lattices $M_1 \subsetneq \cdots \subsetneq M_{i+1}$, where $\pi(L+M) \subsetneq M_1 \subsetneq \cdots \subsetneq M_{i+1} \subsetneq L \cap M$. In particular, A has at most $n-3$ vertices.*

Proof. We proceed by induction on i. If $i = 0$, then A adjacent to $[L \cap M]$ implies A has a unique representative M_1 such that $\pi(L \cap M) \subsetneq M_1 \subsetneq L \cap M$ by Proposition 1.2. Then by [3, p. 322], either $M_1 \subsetneq \pi(L+M)$ or $M_1 \supsetneq \pi(L+M)$. In the second case, we are done, so assume $M_1 \subsetneq \pi(L+M)$. Then $\pi(L \cap M) \subsetneq M_1 \subsetneq \pi(L+M)$. On the other hand, $\pi(L \cap M) \subsetneq \pi L \subsetneq \pi(L+M)$ and $[\pi(L+M) : \pi(L \cap M)] = q^2$. Since A is adjacent to t, [3, p. 322] implies that either $M_1 \subsetneq \pi L$ or $M_1 \supsetneq \pi L$. Thus, either $\pi(L \cap M) \subsetneq M_1 \subsetneq \pi L \subsetneq \pi(L+M)$ or $\pi(L \cap M) \subsetneq \pi L \subsetneq M_1 \subsetneq \pi(L+M)$, which is impossible given the previous index computation.

Now suppose $0 \leq i \leq n-5$ and that the claim holds for any i-simplex in $\Xi_n^c(t, t')$. Let $A \in \Xi_n^c(t, t')$ be an $(i+1)$-simplex and $x \in A$ a vertex. Then the i-simplex $A - \{x\}$ corresponds to a chain of lattices $M'_1 \subsetneq \cdots \subsetneq M'_{i+1}$ such that $\pi(L+M) \subsetneq M'_1 \subsetneq \cdots \subsetneq M'_{i+1} \subsetneq L \cap M$. By the last paragraph, x has a representative M' such that $\pi(L+M) \subsetneq M' \subsetneq L \cap M$. If $M' \subsetneq M'_1$, set $M_1 = M'$ and $M_j = M'_{j-1}$ for all $2 \leq j \leq i+2$. Otherwise, $M' \supsetneq M'_1$ by [3, p. 322]. Let $j \in \{1, \ldots, i+1\}$ be maximal such that $M' \supsetneq M'_j$. If $j = i+1$, set $M_\ell = M'_\ell$ for all $1 \leq \ell \leq i+1$ and $M_{i+2} = M'$. Setting $M_\ell = M'_\ell$ for all $1 \leq \ell \leq j$, $M_{j+1} = M'$, and $M_\ell = M'_{\ell-1}$ for all $j+2 \leq \ell \leq i+2$ finishes the proof if $j \neq i+1$.

Finally, note that if the claim holds for $i \geq n-3$, then A corresponds to a chain of lattices $M_1 \subsetneq \cdots \subsetneq M_{i+1}$, where $\pi(L+M) \subsetneq M_1 \subsetneq \cdots \subsetneq M_{i+1} \subsetneq L \cap M$, contradicting the fact that $[L \cap M : \pi(L+M)] = q^{n-2}$. $\square$

Write $\Xi_n^s(k)$ for the spherical $A_n(k)$ building described in [5, p. 4].

Proposition 1.3. *For any close vertices $t, t' \in \Xi_n$, $\Xi_n^c(t, t')$ is isomorphic (as a poset) to $\Xi_{n-3}^s(k)$ (independent of t and t'), where $\Xi_0^s(k) = \emptyset$.*

Proof. Let $L \in t, M \in t'$ be as in the paragraph preceding Proposition 1.2, and let $\Xi_{n-3}^s(k)$ be the spherical $A_{n-3}(k)$ building with simplices the empty set, together with the nested sequences of non-trivial, proper k-subspaces of $(L \cap M)/\pi(L+M)$. Then by the Correspondence Theorem and the last lemma, there is a bijection between the i-simplices in $\Xi_n^c(t, t')$ and the i-simplices in $\Xi_{n-3}^s(k)$ for all i. Since this bijection preserves the partial order (face) relation, it is a poset isomorphism. $\square$

Theorem 1.2. *If $t, t' \in \Xi_n$ are close vertices, then $m(t, t') = r_{n-2}$ (independent of t and t'). In particular, $\omega_n = (r_n \cdot q)/r_{n-2}$.*

Proof. By the last proposition and previous comments, $m(t, t')$ is the number of chambers in $\Xi_{n-3}^s(k)$. The proof now follows from [6, Proposition 2.4]. $\square$

2 Close Vertices in the Affine Building Δ_n of $\mathrm{Sp}_n(K)$

Let Δ_n denote the affine building naturally associated to $\mathrm{Sp}_n(K)$. Then Δ_n is a subcomplex of Ξ_{2n}, and there is a natural embedding of Δ_n in Ξ_{2n}. As we will see, this embedding allows us to derive information about Δ_n and to prove results for Δ_n by adapting the proofs of the analogous results for Ξ_{2n}. As noted in the introduction, while all the vertices in Ξ_{2n} are special, only two vertices in each chamber in Δ_n are special. Consequently, the Sp_n case requires more care than that needed in the last section. We start by looking at properties of Δ_n that we need to consider close vertices in Δ_n.

2.1 The building Δ_n

The building Δ_n can be modeled as an n-dimensional simplicial complex as follows (see [3, pp. 336 – 337]). Fix a $2n$-dimensional K-vector space V endowed with a non-degenerate, alternating bilinear form $\langle \cdot, \cdot \rangle$, and recall that a subspace U of V is *totally isotropic* if $\langle u, u' \rangle = 0$ for all $u, u' \in U$. A lattice L in V is *primitive* if $\langle L, L \rangle \subseteq \mathcal{O}$ and $\langle \cdot, \cdot \rangle$ induces a non-degenerate, alternating k-bilinear form on $L/\pi L$. Then a *vertex* in Δ_n is a homothety class of lattices in V with a representative L such that there is a primitive lattice L_0 with $\langle L, L \rangle \subseteq \pi\mathcal{O}$ and $\pi L_0 \subseteq L \subseteq L_0$; equivalently, $L/\pi L_0$ is a totally isotropic k-subspace of $L_0/\pi L_0$. Two vertices $t, t' \in \Delta_n$ are *incident* if there are representatives $L \in t$ and $L' \in t'$ such that there is a primitive lattice L_0 with $\langle L, L \rangle \subseteq \pi\mathcal{O}$, $\langle L', L' \rangle \subseteq \pi\mathcal{O}$, and either $\pi L_0 \subseteq L \subseteq L' \subseteq L_0$ or $\pi L_0 \subseteq L' \subseteq L \subseteq L_0$. Thus, a *chamber* in Δ_n has $n+1$ vertices $t_0, \ldots, t_n$ with representatives $L_i \in t_i$ such that L_0 is primitive, $\langle L_i, L_i \rangle \subseteq \pi\mathcal{O}$ for all $1 \leq i \leq n$, and $\pi L_0 \subsetneq L_1 \subsetneq \cdots \subsetneq L_n \subsetneq L_0$. From now on, write that a chamber in

Δ_n corresponds to the chain $\pi L_0 \subsetneq L_1 \subsetneq \cdots \subsetneq L_n \subsetneq L_0$ only when the lattices $L_0, \ldots, L_n$ satisfy the conditions in the last sentence.

Recall that a basis $\{u_1, \ldots, u_n, w_1, \ldots, w_n\}$ for V is *symplectic* if $\langle u_i, w_j \rangle = \delta_{ij}$ (Kronecker delta) and $\langle u_i, u_j \rangle = 0 = \langle w_i, w_j \rangle$ for all i, j. If a 2-dimensional, totally isotropic subspace U of V is a hyperbolic plane, then a frame is an unordered n-tuple $\{\lambda_1^1, \lambda_1^2\}, \ldots, \{\lambda_n^1, \lambda_n^2\}$ of pairs of lines (1-dimensional K-subspaces) in V such that

1. $\lambda_i^1 + \lambda_i^2$ is a hyperbolic plane for all $1 \leq i \leq n$,
2. $\lambda_i^1 + \lambda_i^2$ is orthogonal to $\lambda_j^1 + \lambda_j^2$ for all $i \neq j$, and
3. $V = (\lambda_1^1 + \lambda_1^2) + \cdots + (\lambda_n^1 + \lambda_n^2)$.

A vertex $t \in \Delta_n$ lies in the apartment specified by the frame $\{\lambda_1^1, \lambda_1^2\}, \ldots, \{\lambda_n^1, \lambda_n^2\}$ if for any representative $L \in t$, there are lattices M_i^j in λ_i^j for all i, j such that $L = M_1^1 + M_1^2 + \cdots + M_n^1 + M_n^2$. The following lemma is easily established.

Lemma 2.1.

1. *Every symplectic basis for V specifies an apartment of Δ_n.*
2. *If Σ is an apartment of Δ_n, there is a symplectic basis $\{u_1, \ldots, u_n, w_1, \ldots, w_n\}$ for V such that every vertex in Σ has the form*

$$[\mathcal{O}\pi^{a_1}u_1 + \cdots + \mathcal{O}\pi^{a_n}u_n + \mathcal{O}\pi^{b_1}w_1 + \cdots + \mathcal{O}\pi^{b_n}w_n]$$

for some $a_i, b_i \in \mathbb{Z}$.

Remark. A frame specifying an apartment of Δ_n also specifies an apartment of Ξ_{2n} (see [3, p. 323]). In particular, a symplectic basis for V specifies an apartment of Ξ_{2n}.

Since π is fixed, if $\mathcal{B} = \{u_1, \ldots, u_n, w_1, \ldots, w_n\}$ is a symplectic basis for V, follow [7, p. 3411] and write $(a_1, \ldots, a_n; b_1, \ldots, b_n)_{\mathcal{B}}$ for the lattice $\mathcal{O}\pi^{a_1}u_1 + \cdots + \mathcal{O}\pi^{a_n}u_n + \mathcal{O}\pi^{b_1}w_1 + \cdots + \mathcal{O}\pi^{b_n}w_n$ and $[a_1, \ldots, a_n; b_1, \ldots, b_n]_{\mathcal{B}}$ for its homothety class. Then the lattice $L = (a_1, \ldots, a_n; b_1, \ldots, b_n)_{\mathcal{B}}$ is primitive if and only if $a_i + b_i = 0$ for all i by [7, p. 3411], and $[L]$ is a *special* vertex in Δ_n if and only if $a_i + b_i = \mu$ is constant for all i by [7, Corollary 3.4]. Note that by [7, p. 3412], a chamber in Δ_n has exactly two special vertices.

Lemma 2.2. *Let $t \in \Delta_n$ be a vertex with a primitive representative L, and let Σ be an apartment of Δ_n containing t. Then there is a symplectic basis $\mathcal{B}$ for V specifying Σ as in Lemma 2.1 such that $L = (0, \ldots, 0; 0, \ldots, 0)_{\mathcal{B}}$.*

Proof. This follows from Lemma 2.1 and [7, p. 3411]. □

Let $t \in \Delta_n$ be a vertex. Then the link of t in Δ_n, denoted $\mathrm{lk}_{\Delta_n} t$, is a building (see [1, Proposition IV.1.3]) that is isomorphic (as a poset) to the subposet of Δ_n consisting of those simplices containing t by [1, p. 31]. In particular, if $A \in \Delta_n$ is a codimension-one simplex containing t and $A' \in \mathrm{lk}_{\Delta_n} t$ is the codimension-one simplex corresponding to A, then the number of chambers in Δ_n containing A is the number of chambers in $\mathrm{lk}_{\Delta_n} t$ containing A'. Note that if t is *special*, then [8, p. 35] implies $\mathrm{lk}_{\Delta_n} t$ is isomorphic to the spherical $C_n(k)$ building $\Delta_n^s(k)$ described in [5, pp. 5 – 6].

Proposition 2.1. *Every special vertex in Δ_n is contained in exactly $r(\Delta_n) = \prod_{m=1}^{n} ((q^{2m} - 1)/(q-1))$ chambers in Δ_n.*

Proof. Let $t \in \Delta_n$ be a special vertex. By the preceding comments and [5, pp. 5 – 6], it suffices to count the number of maximal flags of non-trivial, totally isotropic subspaces of a $2n$-dimensional k-vector space endowed with a non-degenerate, alternating bilinear form. An obvious modification of the proof of [6, Proposition 2.4] finishes the proof. $\square$

Remark. The number $r(\Delta_n)$ in the last proposition corresponds to the number r_n given in [6, Proposition 2.4]. Since $\mathrm{Sp}_1(K) = \mathrm{SL}_2(K)$, set $r(\Delta_1) = q+1$ for completeness.

Proposition 2.2. *If $A \in \Delta_n$ is a codimension-one simplex, then A is contained in exactly $q+1$ chambers in Δ_n.*

Proof. Let t be a special vertex in A and A' the codimension-one simplex in $\mathrm{lk}_{\Delta_n} t$ corresponding to A. By the comments preceding the last proposition, it suffices to count the number of chambers in $\Delta_n^s(k)$ containing A'. A case-by-case analysis finishes the proof. $\square$

We now use the fact that Δ_n is a subcomplex of Ξ_{2n} to derive information about Δ_n. For a vertex $t \in \Xi_{2n}$ with representative $L = \mathcal{O}v_1 + \cdots + \mathcal{O}v_{2n}$ and $g \in \mathrm{GL}_{2n}(K)$, define $gt = [\mathcal{O}(gv_1) + \cdots + \mathcal{O}(gv_{2n})]$. Then $\mathrm{GL}_{2n}(K)$ acts transitively on the lattices in V.

Let

$$J_n = \left(\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix}\right) \text{ and } \mathrm{GSp}_n(K) = \left\{ g \in M_{2n}(K) : g^t J_n g = \nu(g) J_n \text{ for some } \nu(g) \in K^\times \right\},$$

so that $\mathrm{Sp}_n(K)$ consists of the matrices $g \in \mathrm{GSp}_n(K)$ with $\nu(g) = 1$. Alternatively, abuse notation and think of $\mathrm{GSp}_n(K)$ as

$$\{g \in \mathrm{GL}_K(V) : \forall\, v_1, v_2 \in V,\ \exists\, \nu(g) \in K^\times \text{ such that } \langle gv_1, gv_2\rangle = \nu(g)\langle v_1, v_2\rangle\}.$$

If $g \in \mathrm{GL}_{2n}(K)$ and $\mathcal{B} = \{v_1, \ldots, v_{2n}\}$ is a basis for V, write $g\mathcal{B}$ for $\{gv_1, \ldots, gv_{2n}\}$.

Lemma 2.3. *The group $\mathrm{Sp}_n(K)$ acts on the set of primitive lattices in V.*

Proof. Let L be a primitive lattice in V, and let Σ be an apartment of Δ_n containing $[L]$ and $\mathcal{B}$ a symplectic basis for V specifying Σ as in Lemma 2.1. Then $L = (a_1, \ldots, a_n; -a_1, \ldots, -a_n)_{\mathcal{B}}$ by [7, p. 3411]; hence, for $g \in \mathrm{Sp}_n(K)$, $g\mathcal{B}$ a symplectic basis for V implies that gL is primitive. $\square$

For the rest of this section, let $\mathcal{B}_0 = \{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ be the standard symplectic basis for V ($f_i = e_{n+i}$ for all i), $L_0 = (0, \ldots, 0; 0, \ldots, 0)_{\mathcal{B}_0}$, and $t_0 = [L_0]$. Following [5, p. 116], assign *types* to the vertices in Ξ_{2n} as follows: assign type 0 to t_0 and type $\mathrm{ord}(\det g) \bmod 2n$ to any other vertex $t = [L] \in \Xi_{2n}$, where $g \in \mathrm{GL}_{2n}(K)$ such that $L = gL_0$. This induces a labelling on the vertices in Δ_n. For the rest of this section, let C_0 be the chamber in Δ_n whose vertices are the homothety classes of the lattices

$$L_0 = (0, \ldots, 0; 0, \ldots, 0)_{\mathcal{B}_0}, L_1 = (0, 1, \ldots, 1; 1, \ldots, 1)_{\mathcal{B}_0}, \ldots, L_n = (0, \ldots, 0; 1, \ldots, 1)_{\mathcal{B}_0}. \quad (3)$$

Note that $[L_i]$ has type $2n - i$ for all $1 \le i \le n$. Recall that since Δ_n is the affine building naturally associated to $\mathrm{Sp}_n(K)$, $\mathrm{Sp}_n(K)$ acts on the vertices in Δ_n in a type-preserving manner and also acts transitively on the chambers in Δ_n.

Proposition 2.3. *If $t \in \Delta_n$ is a vertex, then t has type i for some $i \equiv n, \ldots, 2n \mod 2n$.*

Proof. By the preceding comments, it suffices to show that for all $0 \leq j \leq n$, $[L_j]$ (as in (3)) has type i for some $i \equiv n, \ldots, 2n \mod 2n$, which we already observed. $\square$

We now use types to characterize the vertices in Δ_n with a primitive representative, as well as those that are special.

Proposition 2.4. *A vertex in Δ_n has a primitive representative if and only if it has type* 0.

Proof. Let $t \in \Delta_n$ be a type 0 vertex and $C \in \Delta_n$ a chamber containing t. Choose $g \in \mathrm{Sp}_n(K)$ such that $gC_0 = C$. Then $gL_0 \in t$. Since L_0 is primitive, Lemma 2.3 implies that gL_0 is primitive. Conversely, let $t \in \Delta_n$ be a vertex with a primitive representative L, and let $C \in \Delta_n$ be a chamber containing t. Let $g \in \mathrm{Sp}_n(K)$ such that $gC = C_0$. Then $gL = \pi^m L_j$ for some $0 \leq j \leq n$ and some $m \in \mathbb{Z}$. If $L_j = (a_1, \ldots, a_n; b_1, \ldots, b_n)_{\mathcal{B}_0}$ as in (3), then $gL = (a_1 + m, \ldots, a_n + m; b_1 + m, \ldots, b_n + m)_{\mathcal{B}_0}$. But gL primitive (by Lemma 2.3) implies that $a_i + b_i = -2m$ for all i. By (3), $m = 0$ and $gt = [L_0]$; hence, t has type 0. $\square$

Proposition 2.5. *A vertex in Δ_n is special if and only if it has type 0 or n.*

Proof. Let $t \in \Delta_n$ be a type 0 (resp., type n) vertex, and let $C \in \Delta_n$ be a chamber containing t. If $g \in \mathrm{Sp}_n(K)$ such that $gC_0 = C$, then $t = g[L_0]$ (resp., $t = g[L_n]$), and t is special by [7, Corollary 3.4]. Conversely, let $t \in \Delta_n$ be a special vertex. Let $C \in \Delta_n$ be a chamber containing t, Σ an apartment of Δ_n containing C, and $\mathcal{B}$ a symplectic basis for V specifying Σ as in Lemma 2.1. By [7, Corollary 3.4], $t = [a_1, \ldots, a_n; \mu - a_1, \ldots, \mu - a_n]_{\mathcal{B}}$ for some $\mu \in \mathbb{Z}$. If $g \in \mathrm{Sp}_n(K)$ such that $gC = C_0$, then $gt = [L_i]$ for some $0 \leq i \leq n$; hence, gt special, [7, Corollary 3.4], and (3) imply $i = 0$ or $i = n$, and t has type 0 or n. $\square$

We now consider the action of $\mathrm{GSp}_n(K)$ on the vertices in Ξ_{2n}.

Proposition 2.6. *If $[L]$ is a type i vertex in Ξ_{2n}, then for any $g \in \mathrm{GL}_{2n}(K)$, the vertex $g[L] \in \Xi_{2n}$ has type $i + \mathrm{ord}(\det g) \mod 2n$.*

Proof. Since $[L]$ has type i, we can write $L = g_i L_0$, where $g_i \in \mathrm{GL}_{2n}(K)$ with $\mathrm{ord}(\det g_i) \equiv i \mod 2n$. Then $g[L]$ has type $\mathrm{ord}(\det(gg_i)) \mod 2n \equiv i + \mathrm{ord}(\det g) \mod 2n$. $\square$

Corollary 2.1. *If $g \in \mathrm{GSp}_n(K)$ with $\mathrm{ord}(\nu(g)) \equiv 1 \mod 2$, then g maps a non-special vertex in Δ_n to a vertex in Ξ_{2n} that is not in Δ_n.*

Proof. First note that $g \in \mathrm{GSp}_n(K)$ with $\mathrm{ord}(\nu(g)) \equiv 1 \mod 2$ implies $\mathrm{ord}(\det g) \equiv n \mod 2n$. If t is a non-special vertex in Δ_n, then t has type i for some $n + 1 \leq i \leq 2n - 1$ by Propositions 2.3 and 2.5. Thus, the last proposition implies gt has type $i + n \mod 2n \in \{1, \ldots, n - 1\}$. Proposition 2.3 finishes the proof. $\square$

2.2 The building Δ_n in the building Ξ_{2n}

Let $C \in \Delta_n$ be a chamber corresponding to the chain $\pi L_0 \subsetneq L_1 \subsetneq \cdots \subsetneq L_n \subsetneq L_0$. Let Σ be an apartment of Δ_n containing C, $\mathcal{B}$ a symplectic basis for V specifying Σ as in Lemma 2.1, and $\widetilde{\Sigma}$ the apartment of Ξ_{2n} specified by $\mathcal{B}$. Let $D \in \widetilde{\Sigma}$ be any chamber containing C. Then D corresponds to the chain $\pi L_0 \subsetneq L_1 \subsetneq \cdots \subsetneq L_n \subsetneq L_{n+1} \subsetneq \cdots \subsetneq L_{2n-1} \subsetneq L_0$ for some lattices $L_{n+1}, \ldots, L_{2n-1}$ in V. For $0 \le j \le 2n-1$, write

$$L_j = (a_1^{(j)}, \ldots, a_n^{(j)}; b_1^{(j)}, \ldots, b_n^{(j)})_{\mathcal{B}}.$$

Lemma 2.4. *The two special vertices in C are $[L_0]$ and $[L_n]$.*

Proof. The fact that $[L_0]$ is special follows from [7, Corollary 3.4] and [7, p. 3411]. To see that $[L_n]$ is special, note that if L_j represents a special vertex in C for $1 \le j \le n$, then $a_i^{(j)} + b_i^{(j)} = \mu$ for all i (by [7, Corollary 3.4]), where $\mu \in \{1, 2\}$ (since $\langle L_j, L_j\rangle \subseteq \pi\mathcal{O}$). But $\mu = 2$ implies $L_j = \pi L_0$, which is impossible. Thus, $a_i^{(j)} + b_i^{(j)} = 1$ for all i and $L_j/\pi L_0 \cong k^n$; hence, $j = n$. □

For $\mathcal{B} = \{u_1, \ldots, u_n, w_1, \ldots, w_n\}$ a symplectic basis for V and $g \in \mathrm{GSp}_n(K)$, let

$$\mathcal{B}_g := \{\nu(g)^{-1} g u_1, \ldots, \nu(g)^{-1} g u_n, g w_1, \ldots, g w_n\}.$$

Note that $\mathcal{B}_g$ is a symplectic basis for V; hence, $L = (a_1, \ldots, a_n; b_1, \ldots, b_n)_{\mathcal{B}}$ and $\mathrm{ord}(\nu(g)) = m$ imply $gL = (a_1 + m, \ldots, a_n + m; b_1, \ldots, b_n)_{\mathcal{B}_g}$.

Proposition 2.7. *The group $\mathrm{GSp}_n(K)$ acts transitively on the special vertices in Δ_n.*

Proof. Note that if $\mathrm{GSp}_n(K)$ acts on the special vertices in Δ_n, then [7, Proposition 3.3] implies that the action is transitive. We thus show that $\mathrm{GSp}_n(K)$ acts on the special vertices in Δ_n. Let $t \in \Delta_n$ be a special vertex and $L \in t$ a representative such that there is a primitive lattice L_0 with $\langle L, L\rangle \subseteq \pi\mathcal{O}$ and $\pi L_0 \subseteq L \subseteq L_0$. Let Σ be an apartment of Δ_n containing t and $[L_0]$, and let $\mathcal{B}$ be a symplectic basis for V specifying Σ as in Lemma 2.1. Then [7, p. 3411], the last lemma, and [7, Corollary 3.4] imply

$$L_0 = (c_1, \ldots, c_n; -c_1, \ldots, -c_n)_{\mathcal{B}} \qquad \text{and} \qquad L = (a_1, \ldots, a_n; \mu - a_1, \ldots, \mu - a_n)_{\mathcal{B}},$$

where $\mu \in \{1, 2\}$. Let $g \in \mathrm{GSp}_n(K)$ with $\mathrm{ord}(\nu(g)) = m$. Since $gt = [a_1 + m, \ldots, a_n + m; \mu - a_1, \ldots, \mu - a_n]_{\mathcal{B}_g}$, [7, Corollary 3.4] implies that it suffices to show gt is a vertex in Δ_n. First suppose $m \equiv 0 \mod 2$, say $m = 2r$. Then $\pi^{-r} g L_0$ is primitive, $\langle \pi^{-r} gL, \pi^{-r} gL\rangle \subseteq \pi\mathcal{O}$, and $\pi^{-r} g(\pi L_0) \subseteq \pi^{-r} gL \subseteq \pi^{-r} g L_0$; i.e., gt is a vertex in Δ_n. Now suppose $m = 2r + 1$. If $\mu = 1$, then $\pi^{-r-1} gL$ is primitive and gt is a vertex in Δ_n. Otherwise, $\mu = 2$, and $\langle \pi^{-r-1} gL, \pi^{-r-1} gL\rangle \subseteq \pi\mathcal{O}$. Let $\pi M_0 = (a_1 + r, \ldots, a_n + r; \mu - a_1 - r, \ldots, \mu - a_n - r)_{\mathcal{B}_g}$. Then M_0 is primitive and $\pi M_0 \subseteq \pi^{-r-1} gL \subseteq M_0$; i.e., gt is a vertex in Δ_n. Thus, $\mathrm{GSp}_n(K)$ acts on the special vertices in Δ_n. □

Note that by Propositions 2.4 and 2.5, $[L_n]$ has type n. Then by Proposition 2.3, the type of $[L_j]$ is in $\{n+1, \ldots, 2n-1\}$ for all $1 \le j \le n-1$ and the type of $[L_i]$ is in $\{1, \ldots, n-1\}$ for all $n+1 \le i \le 2n-1$.

Lemma 2.5. *Let $g \in \mathrm{GSp}_n(K)$ with $\mathrm{ord}(\nu(g)) \equiv 1 \mod 2$. If $L_0, L_n, \ldots, L_{2n-1}$ are lattices in V as above, then the vertices $g[L_n], \ldots, g[L_{2n-1}], g[L_0]$ in Ξ_{2n} are the vertices in a chamber in Δ_n.*

Proof. Write $\mathrm{ord}(\nu(g)) = 2r+1$. Then Lemma 2.4 and [7, p. 3411] imply that $L_n' = \pi^{-(r+1)} g L_n$ is primitive (see the proof of Proposition 2.7). Furthermore, if $L_j' = \pi^{-r} g L_j$ for $j = 0, n+1, \ldots, 2n-1$, then $\pi L_n' \subsetneq L_{n+1}' \subsetneq \cdots \subsetneq L_{2n-1}' \subsetneq L_0' \subsetneq L_n'$ and $\langle L_j', L_j' \rangle \subseteq \pi\mathcal{O}$ for $j = 0, n+1, \ldots, 2n-1$; i.e., $[L_n'], \ldots, [L_{2n-1}'], [L_0']$ are the vertices in a chamber in Δ_n. $\square$

Lemma 2.6. *Let Σ be an apartment of Δ_n and $\widetilde{\Sigma}$ the apartment of Ξ_{2n} such that $\mathcal{B}$ a symplectic basis for V specifying Σ implies $\mathcal{B}$ specifies $\widetilde{\Sigma}$. If C, C' is a gallery in Σ, then there is a gallery D, D' in $\widetilde{\Sigma}$ such that D (resp., D') contains C (resp., C') and $C \neq C'$ implies $D \neq D'$.*

Remark. More generally, if $C_0, \ldots, C_m$ is a gallery in Δ_n, then there is a gallery $D_0, \ldots, D_\ell$ in Ξ_{2n} and integers $0 \le i_0 < \cdots < i_m \le \ell$ such that D_j contains C_0 for all $0 \le j \le i_0$ and D_j contains C_r for all $i_{r-1} < j \le i_r$ and all $1 \le r \le m$.

Proof. If $C = C'$, set $D = D'$, where $D \in \widetilde{\Sigma}$ is a chamber containing C. Now suppose $C \neq C'$, with C corresponding to the chain

$$\pi L_0 \subsetneq L_1 \subsetneq \cdots \subsetneq L_n \subsetneq L_0. \tag{4}$$

Let $\mathcal{B}$ be a symplectic basis for V specifying Σ as in Lemma 2.1, and let $0 \le j \le n$ such that $C \cap C'$ corresponds to (4) with L_j deleted if $1 \le j \le n$ or with both πL_0 and L_0 deleted if $j = 0$. Note that if t' is the vertex in C' not in C, then t' has a representative L' such that C' corresponds to (4) with L_j replaced by L'.

If $1 \le j \le n-1$, then [7, p. 3411], Lemma 2.4, and (4) imply $L_0 = (a_1, \ldots, a_n; -a_1, \ldots, -a_n)_{\mathcal{B}}$ and $L_n = (b_1, \ldots, b_n; 1-b_1, \ldots, 1-b_n)_{\mathcal{B}}$, where $a_i + 1 \ge b_i \ge a_i$ for all i. For $1 \le i \le n$, let $a_{n+i} = -a_i$ and $b_{n+i} = 1 - b_i$. Let $\{i_1, \ldots, i_n\}$ be the n values of i such that $b_i = a_i + 1$, and for $1 \le r \le n-1$, set $L_{n+r} = (c_1, \ldots, c_n; c_{n+1}, \ldots, c_{2n})_{\mathcal{B}}$, where $c_\ell = b_\ell - 1 = a_\ell$ if $\ell \in \{i_1, \ldots, i_r\}$ and $c_\ell = b_\ell$ otherwise. Then $L_n \subsetneq L_{n+1} \subsetneq \cdots \subsetneq L_{2n-1} \subsetneq L_0$, and letting $D \in \widetilde{\Sigma}$ (resp., $D' \in \widetilde{\Sigma}$) be the simplex with vertices the vertices in C (resp., the vertices in C'), together with $[L_{n+1}], \ldots, [L_{2n-1}]$ finishes the proof in this case

If $j = n$, write $L_0 = (a_1, \ldots, a_n; -a_1, \ldots, -a_n)_{\mathcal{B}}$, $L_n = (b_1, \ldots, b_n; 1-b_1, \ldots, 1-b_n)_{\mathcal{B}}$, and $L' = (b_1', \ldots, b_n'; 1-b_1', \ldots, 1-b_n')_{\mathcal{B}}$. Note that $a_i + 1 \ge b_i, b_i' \ge a_i$ for all i and $b_i \neq b_i'$ for at least one value of i. Let $L_{n+1} = (c_1, \ldots, c_n; c_{n+1}, \ldots, c_{2n})_{\mathcal{B}}$, where $c_i = \min\{b_i, b_i'\}$ and $c_{n+i} = \min\{1-b_i, 1-b_i'\}$ for $1 \le i \le n$. Then $L_{n+1} = L_n + L'$, so $L_n, L' \subsetneq L_{n+1}$ and $[L_{n+1} : L_n] = q = [L_{n+1} : L']$. An obvious modification of the second half of the last paragraph finishes the proof in this case.

Finally, if $j = 0$, write $L_0 = (a_1, \ldots, a_n; -a_1, \ldots, -a_n)_{\mathcal{B}}$, $L' = (a_1', \ldots, a_n'; -a_1', \ldots, -a_n')_{\mathcal{B}}$, and $L_n = (b_1, \ldots, b_n; 1-b_1, \ldots, 1-b_n)_{\mathcal{B}}$. Note that $a_i + 1, a_i' + 1 \ge b_i \ge a_i, a_i'$ for all i and $a_i \neq a_i'$ for at least one value of i. Let $L_{2n-1} = (c_1, \ldots, c_n; c_{n+1}, \ldots, c_{2n})_{\mathcal{B}}$, where $c_i = \max\{a_i, a_i'\}$ and $c_{n+i} = \max\{-a_i, -a_i'\}$ for $1 \le i \le n$. Then $L_{2n-1} = L_0 \cap L'$, so $L_{2n-1} \subsetneq L_0, L'$ and $[L_0 : L_{2n-1}] = q = [L' : L_{2n-1}]$. An obvious modification of the second half of the first paragraph finishes the proof in this case. $\square$

It will turn out to be convenient to first prove results about the type 0 vertices in Δ_n and to then use the transitive action of $\mathrm{GSp}_n(K)$ on the special vertices in Δ_n (see Proposition 2.7) to deduce the same results about the type n vertices in Δ_n. For $g \in \mathrm{GL}_{2n}(K)$ and a chamber $C \in \Xi_{2n}$, abuse notation and write gC for the image of the vertices in C under the action of g.

Proposition 2.8. *The group $\mathrm{GL}_{2n}(K)$ (resp., $\mathrm{GSp}_n(K)$) maps a gallery in Ξ_{2n} of length m to a gallery in Ξ_{2n} of length m. In particular, if $C \neq C'$ are adjacent chambers in Ξ_{2n} and $g \in \mathrm{GL}_{2n}(K)$ (resp., $g \in \mathrm{GSp}_n(K)$), then $gC \neq gC'$ are adjacent chambers in Ξ_{2n}.*

Proof. Let $C_0, \ldots, C_m$ be a gallery in Ξ_{2n}, and let $g \in \mathrm{GL}_{2n}(K)$. If $m = 0$ and C_0 corresponds to the chain $\pi L_0 \subsetneq L_1 \subsetneq \cdots \subsetneq L_{2n-1} \subsetneq L_0$, then $g(\pi L_0) \subsetneq gL_1 \subsetneq \cdots \subsetneq gL_{2n-1} \subsetneq gL_0$; i.e., gC_0 is a chamber in Ξ_{2n}. If $m = 1$ and $C_0 = C_1$, then gC_0, gC_1 is a gallery in Ξ_{2n}, so suppose $C_0 \neq C_1$. Let $t_0, \ldots, t_{2n-1}$ (resp., $x_0, \ldots, x_{2n-1}$) be the vertices in C_0 (resp., in C_1), and let $0 \leq j \leq 2n-1$ such that $t_j \neq x_j$. For $0 \leq i \leq 2n-1$, let $L_i \in t_i$ (resp., let $M_i \in x_i$) such that $\pi L_0 \subsetneq L_1 \subsetneq \cdots \subsetneq L_{2n-1} \subsetneq L_0$ (resp., $\pi M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_{2n-1} \subsetneq M_0$) corresponds to C_0 (resp., to C_1). Then $g(\pi L_0) \subsetneq gL_1 \subsetneq \cdots \subsetneq gL_{2n-1} \subsetneq gL_0$ (resp., $g(\pi M_0) \subsetneq gM_1 \subsetneq \cdots \subsetneq gM_{2n-1} \subsetneq gM_0$). Since $t_i = x_i$ implies $gt_i = gx_i$, gC_0, gC_1 is a gallery in Ξ_{2n}. The fact that $gC_0 \neq gC_1$ follows from the fact that $gx_j \neq gt_j$. The proof for $m \geq 2$ follows from the fact that gC_i, gC_{i+1} is a gallery in Ξ_{2n} for all $0 \leq i \leq m-1$. $\square$

2.3 Counting close vertices in Δ_n

Let $\Gamma = \mathrm{Sp}_n(\mathcal{O})$, and note that the analogues of the results in section 4.1 of [7] hold if $\mathrm{GSp}_n(K)$ acts on the lattices in V on the left (rather than on the right). The following is an analogue of Theorem 3.3 of [6] for the special vertices in Δ_n.

Theorem 2.1. *If $t \in \Delta_n$ is a special vertex, then the number of vertices in Δ_n close to t is the number of left cosets of Γ in*

$$\Gamma \mathrm{diag}(1, \underbrace{\pi, \ldots, \pi}_{n-1}, \pi^2, \pi, \ldots, \pi)\Gamma.$$

Proof. First note that by Proposition 2.5, a special vertex in Δ_n has type either 0 or n. Let $t \in \Delta_n$ be a special vertex and $t' \in \Delta_n$ a vertex close to t. Then there are adjacent chambers $C, C' \in \Delta_n$ such that $t \in C$, $t' \in C'$, but $t, t' \notin C \cap C'$. Let Σ be an apartment of Δ_n containing C and C'. If t has type 0, then by Lemma 2.2, we may assume that relative to some symplectic basis $\mathcal{B}$ for V specifying Σ, $t = [0, \ldots, 0; 0, \ldots, 0]_{\mathcal{B}} \in C_0$, where $C_0 \in \Sigma$ is the chamber with vertices $[0, \ldots, 0; 0, \ldots, 0]_{\mathcal{B}}, [0, 1, \ldots, 1; 1, \ldots, 1]_{\mathcal{B}}, \ldots, [0, \ldots, 0; 1, \ldots, 1]_{\mathcal{B}}$. A straightforward modification of the fourth and fifth paragraphs of the proof of [6, Theorem 3.3] using the reflections defined in [7, p. 3411] finishes the proof in this case.

Now suppose t has type n, and let $\mathcal{B}$ be a symplectic basis for V specifying Σ as in Lemma 2.1. Let $\widetilde{\Sigma}$ be the apartment of Ξ_{2n} specified by $\mathcal{B}$, and let $D, D' \in \widetilde{\Sigma}$ be adjacent chambers with C in D, C' in D', and $D \neq D'$ as in Lemma 2.6. Let $g \in \mathrm{GSp}_n(K)$ with $\mathrm{ord}(\nu(g)) \equiv 1 \mod 2$. Then by Proposition 2.6, gt has type 0. By Lemma 2.5, gD (resp., gD') contains a chamber $C_1 \in \Delta_n$ (resp., a chamber $C_1' \in \Delta_n$) with $gt \in C_1$ (resp., with $gt' \in C_1'$). Furthermore, $gD \neq gD'$ are adjacent chambers in Ξ_{2n} and $gt, gt' \notin gD \cap gD'$ by

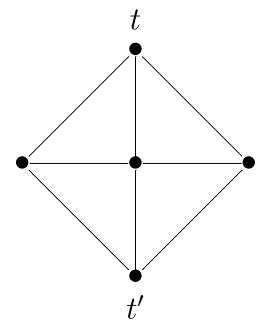


Figure 3: Two close special vertices, both of type 0, in Δ_2.

Proposition 2.8; i.e., gt and gt' are close vertices in Δ_n. Finally, if S_t and S_{gt} are the sets of vertices in Δ_n close to t and gt, respectively, then $\mathrm{Card}(S_t) = \mathrm{Card}(S_{gt})$, and the last paragraph finishes the proof. □

Remark. The analogues of the results in [7, Section 4.1] also hold if $\mathrm{Sp}_n(\mathcal{O})$ and $\mathrm{GSp}_n^S(K)$ are replaced by $\mathrm{GSp}_n(\mathcal{O}) = \mathrm{GL}_{2n}(\mathcal{O}) \cap \mathrm{GSp}_n(K)$ and $\mathrm{GSp}_n(K)$, respectively, and with $\mathrm{GSp}_n(K)$ acting on the left rather than on the right. In addition, the analogue of the above theorem holds with $\Gamma = \mathrm{GSp}_n(\mathcal{O})$; hence, so does Corollary 2.2.

We now count the number of vertices in Δ_n close to a given special vertex $t \in \Delta_n$. By Proposition 2.5 and Theorem 2.1, it suffices to assume t has type 0. By Proposition 2.4, t has a primitive representative L, so a chamber $C \in \Delta_n$ containing t corresponds to a chain of the form

$$\pi L \overset{q}{\subsetneq} L_1 \overset{q}{\subsetneq} \cdots \overset{q}{\subsetneq} L_n \overset{q^n}{\subsetneq} L. \tag{5}$$

The codimension-one face in C not containing t thus corresponds to the chain

$$L_1 \overset{q}{\subsetneq} \cdots \overset{q}{\subsetneq} L_n,$$

and a vertex in Δ_n is close to t if it has a primitive representative $M \neq L$ such that

$$\pi M \overset{q}{\subsetneq} L_1 \overset{q}{\subsetneq} \cdots \overset{q}{\subsetneq} L_n \overset{q^n}{\subsetneq} M. \tag{6}$$

Given the lattice L_1, the possible L and M satisfy $L \neq M \subsetneq \pi^{-1}L_1$ with $[\pi^{-1}L_1 : L] = q = [\pi^{-1}L_1 : M]$ and both L and M primitive. On the other hand, if $t, t' \in \Delta_n$ are close type 0 vertices, then there must be primitive representatives $L \in t$ and $M \in t'$ and lattices $L_1, \ldots, L_n$ as in (5) such that $L \neq M \subsetneq \pi^{-1}L_1$. The same argument as in Section 1 shows that $\pi^{-1}L_1 = L + M$, but we can vary $L_2, \ldots, L_n$ as long as $\langle L_i, L_i \rangle \subseteq \pi\mathcal{O}$ for all $2 \leq i \leq n$ and the chains

$$\pi L \subsetneq L_1 \subsetneq L_2 \subsetneq \cdots \subsetneq L_n \subsetneq L \qquad \text{and} \qquad \pi M \subsetneq L_1 \subsetneq L_2 \subsetneq \cdots \subsetneq L_n \subsetneq M$$

correspond to chambers in Δ_n. In other words (as in the case of Ξ_n), if $t, t' \in \Delta_n$ are close type 0 vertices, there may be more than one pair of adjacent chambers $C, C' \in \Delta_n$ such that $t \in C$, $t' \in C'$, and $t, t' \notin C \cap C'$ (see Figure 3). We return to this later.

Before we count the number of vertices in Δ_n close to t, we make a few observations similar to those preceding Proposition 1.1. Fix a primitive representative $L \in t$. Then

$L/\pi L \cong k^{2n}$ is endowed with a non-degenerate, alternating k-bilinear form. Moreover, the Correspondence Theorem, the fact that any $\mathcal{O}$-submodule of L containing πL is a lattice in V, and the fact that every 1-dimensional k-subspace of $L/\pi L$ is totally isotropic imply that the number of L_1 is the number of 1-dimensional k-subspaces of $L/\pi L$. Given L_1, let $C \in \Delta_n$ be a chamber containing $[L_1]$ and t, and let A be the codimension-one face in C not containing t. Then the number of primitive lattices $M \neq L$ in V such that $M \subsetneq \pi^{-1}L_1$ and $[\pi^{-1}L_1 : M] = q$ is one less than the number of chambers in Δ_n containing A.

Proposition 2.9. *If $t \in \Delta_n$ is a special vertex, then the number $\omega(\Delta_n)$ of vertices in Δ_n close to t is*

$$\frac{q^{2n}-1}{q-1} \cdot q$$

(independent of t).

Proof. This follows from the preceding comments, the fact that the number of 1-dimensional subspaces of $\mathbb{F}_q^m$ is exactly $(q^m-1)/(q-1)$, and Proposition 2.2. □

Corollary 2.2. *The number of left cosets of $\Gamma = \mathrm{Sp}_n(\mathcal{O})$ in*

$$\Gamma \mathrm{diag}(1, \underbrace{\pi, \ldots, \pi}_{n-1}, \pi^2, \pi, \ldots, \pi)\Gamma$$

is $((q^{2n}-1)\cdot q)/(q-1)$.

Proof. This follows from Theorem 2.1 and the last proposition. □

Proposition 2.1 and the last proposition prove the following analogue of Theorem 1.1.

Theorem 2.2. *Let $r(\Delta_n)$ be the number of chambers in Δ_n containing a given special vertex (as in Proposition 2.1) and $\omega(\Delta_n)$ the number of vertices in Δ_n close to a given special vertex in Δ_n (as in Proposition 2.9). Then for all $n \geq 2$, $q \cdot r(\Delta_n) = r(\Delta_{n-1})\,\omega(\Delta_n)$, where $r(\Delta_1) = q+1$.*

When the given vertex in Δ_n has type 0, we can also give a combinatorial proof of Theorem 2.2. As in Section 1, if $t \in \Delta_n$ is a fixed type 0 vertex, then we can try to count the number of vertices in Δ_n close to t by counting the number of galleries (in Δ_n) of length 1 starting at a chamber containing t and ending at a chamber not containing t. An argument analogous to that in Section 1 shows that if $t' \in \Delta_n$ is a vertex close to t, then $\omega(\Delta_n) = (r(\Delta_n)\cdot q)/m(\Delta_n, t, t')$, where $m(\Delta_n, t, t')$ is the number of galleries of length 1 in Δ_n whose initial chamber contains t and whose ending chamber contains t'.

To determine $m(\Delta_n, t, t')$, fix the following notation for the rest of this section. For close special vertices $t, t' \in \Delta_n$ with t of type 0, let $L \in t$, $M \in t'$ be primitive representatives (by Proposition 2.4) such that there are lattices $L_1, \ldots, L_n$ as in (5) and (6) with $\langle L_i, L_i \rangle \subseteq \pi\mathcal{O}$ for all $1 \leq i \leq n$. Recall that $L_1 = \pi(L+M)$, but we can vary $L_2, \ldots, L_n$ as long as $\langle L_i, L_i \rangle \subseteq \pi\mathcal{O}$ for all $2 \leq i \leq n$ and the chains

$$\pi L \subsetneq L_1 \subsetneq L_2 \subsetneq \cdots \subsetneq L_n \subsetneq L \qquad \text{and} \qquad \pi M \subsetneq L_1 \subsetneq L_2 \subsetneq \cdots \subsetneq L_n \subsetneq M$$

correspond to chambers in Δ_n. As in Section 1, each gallery in Δ_n counted by $m(\Delta_n, t, t')$ is uniquely determined by $L_2, \ldots, L_n$. Define two vertices in Δ_n to be *adjacent* if they are distinct and incident.

Lemma 2.7. *Let $t, t' \in \Delta_n$ be adjacent vertices such that t has a primitive representative L. Then t' has a unique representative L' such that $\langle L', L'\rangle \subseteq \pi\mathcal{O}$ and $\pi L \subsetneq L' \subsetneq L$.*

Proof. Since t and t' are adjacent vertices in Ξ_{2n}, by Proposition 1.2, t' has a unique representative L' such that $\pi L \subsetneq L' \subsetneq L$. It thus suffices to show that $\langle L', L'\rangle \subseteq \pi\mathcal{O}$. But t and t' incident vertices in Δ_n with $t \neq t'$ implies they have representatives $M \in t$ and $M' \in t'$ such that there is a primitive lattice L_0 with $\langle M, M\rangle \subseteq \pi\mathcal{O}$, $\langle M', M'\rangle \subseteq \pi\mathcal{O}$, and either $\pi L_0 \subseteq M \subsetneq M' \subseteq L_0$ or $\pi L_0 \subseteq M' \subsetneq M \subseteq L_0$. Suppose $\pi L_0 \subseteq M \subsetneq M' \subseteq L_0$ (resp., $\pi L_0 \subseteq M' \subsetneq M \subseteq L_0$). Then M and πL (resp., M and L) homothetic implies $\pi L = \pi^r M$ (resp., $L = \pi^r M$) for some $r \in \mathbb{Z}$; hence, $\pi L \subsetneq \pi^r M' \subsetneq L$. Let $L' = \pi^r M'$. Since L is primitive, $\langle \pi^{r-1}M, \pi^{r-1}M\rangle \subseteq \mathcal{O}$ (resp., $\langle \pi^r M, \pi^r M\rangle \subseteq \mathcal{O}$). On the other hand, $\langle \pi^{r-1}M, \pi^{r-1}M\rangle \subseteq \pi^{2(r-1)+1}\mathcal{O}$ (resp., $\langle \pi^r M, \pi^r M\rangle \subseteq \pi^{2r+1}\mathcal{O}$), so $r \in \mathbb{Z}^+$ (resp., $r \in \mathbb{Z}^{\geq 0}$) and $\langle L', L'\rangle \subseteq \pi\mathcal{O}$. ☐

Consider the set of vertices in Δ_n that are adjacent to t, t', and $[L+M]$, and define two such vertices to be incident if they are incident as vertices in Δ_n. Let $\Delta_n^c(t,t')$ be the set consisting of

- the empty set,
- all vertices in Δ_n adjacent to t, t', and $[L+M]$, and
- all finite sets A of vertices in Δ_n adjacent to t, t', and $[L+M]$ such that any two vertices in A are adjacent.

Then $\Delta_n^c(t,t')$ is a simplicial complex. In particular, $\Delta_n^c(t,t')$ is a subcomplex of Δ_n.

Lemma 2.8. *If $\emptyset \neq A \in \Delta_n^c(t,t')$ is an i-simplex, then A corresponds to a chain of lattices $M_1 \subsetneq \cdots \subsetneq M_{i+1}$, where $\langle M_j, M_j\rangle \subseteq \pi\mathcal{O}$ for all $1 \leq j \leq i+1$ and $\pi(L+M) \subsetneq M_1 \subsetneq \cdots \subsetneq M_{i+1} \subsetneq L \cap M$. In particular, A has at most $n-1$ vertices.*

Proof. As in the proof of Lemma 1.1, we proceed by induction on i. If $i = 0$, then L primitive, A adjacent to t, and Lemma 2.7 imply A has a unique representative M_1 such that $\langle M_1, M_1\rangle \subseteq \pi\mathcal{O}$ and $\pi L \subsetneq M_1 \subsetneq L$. Since A and $[L+M]$ are adjacent vertices in Ξ_{2n}, either $M_1 \subsetneq \pi(L+M)$ or $M_1 \supsetneq \pi(L+M)$ by [3, p. 322]. But $M_1 \subsetneq \pi(L+M)$ means $\pi L \subsetneq M_1 \subsetneq \pi(L+M)$, which is impossible since $[\pi(L+M) : \pi L] = q$; hence, $M_1 \supsetneq \pi(L+M)$. Then A and t' adjacent vertices in Ξ_{2n} and [3, p. 322] imply that either $M_1 \subsetneq M$ or $M_1 \supsetneq M$. Since $M_1 \supsetneq M$ means $M \subsetneq M_1 \subsetneq L$, which contradicts the fact that $[M : \pi(L+M)] = [L : \pi(L+M)]$, $M_1 \subsetneq M$ and $M_1 \subseteq L \cap M$. Moreover, $\langle M_1, M_1\rangle \subseteq \pi\mathcal{O}$ implies $M_1/\pi L$ is a totally isotropic k-subspace of $L/\pi L$ and $[M_1 : \pi L] \leq q^n$. The fact that $[L \cap M : \pi L] = q^{2n-1}$ finishes the proof in this case.

Recall that $\langle \cdot, \cdot \rangle$ induces a non-degenerate, alternating k-bilinear form on $L/\pi L$. Then with respect to this induced bilinear form, $(L \cap M)/\pi L$ is the orthogonal complement of $\pi(L+M)/\pi L$ in $L/\pi L$. In addition, $\langle \cdot, \cdot \rangle$ induces a non-degenerate, alternating k-bilinear form on $(L \cap M)/\pi(L+M) \cong k^{2(n-1)}$, and there is a bijection between nested sequences $S_1 \subsetneq \cdots \subsetneq S_{i+1}$ of totally isotropic k-subspaces of $(L \cap M)/\pi(L+M)$ and chains of $\mathcal{O}$-submodules $M_1 \subsetneq \cdots \subsetneq M_{i+1}$ of $L \cap M$ containing $\pi(L+M)$ with $\langle M_j, M_j\rangle \subseteq \pi\mathcal{O}$ for all $1 \leq j \leq i+1$. An obvious modification of the second paragraph of the proof of Lemma 1.1 finishes the proof. ☐

Recall that $\Delta_n^s(k)$ denotes the spherical $C_n(k)$ building described in [5, pp. 5 – 6].

Proposition 2.10. *For any close special vertices $t, t' \in \Delta_n$ with t of type 0, $\Delta_n^c(t, t')$ is isomorphic (as a poset) to $\Delta_{n-1}^s(k)$ (independent of t and t' with t of type 0).*

Proof. Let $L \in t, M \in t'$ be primitive representatives as in the paragraph preceding Lemma 2.7, and let $\Delta_{n-1}^s(k)$ be the spherical $C_{n-1}(k)$ building with simplices the empty set, together with the nested sequences of non-trivial, totally isotropic k-subspaces of $(L \cap M)/\pi(L+M)$. Then the last lemma implies that there is a bijection between the i-simplices in $\Delta_n^c(t, t')$ and the i-simplices in $\Delta_{n-1}^s(k)$ for all i. Since this bijection preserves the partial order (face) relation, it is a poset isomorphism. □

Proposition 2.11. *If $t, t' \in \Delta_n$ are close special vertices with t of type 0, then $m(\Delta_n, t, t') = r(\Delta_{n-1})$ (independent of t and t'). In particular, $\omega(\Delta_n) = (r(\Delta_n) \cdot q)/r(\Delta_{n-1})$.*

Proof. The proof is an obvious modification of the proof of Theorem 1.2. □

References

[1] K. Brown. *Buildings*. Springer-Verlag, New York, 1989.

[2] D. Cartwright. Harmonic functions on buildings of type $\widetilde{A}_n$. In M. Picardello and W. Woess, editors, *Random Walks and Discrete Potential Theory*, pages 104 – 138, Cambridge, 1999. Cambridge University Press.

[3] P. Garrett. *Buildings and Classical Groups*. Chapman & Hall, London, 1997.

[4] J. Parkinson. Buildings and Hecke algebras. *Journal of Algebra*, 297(1):1 – 49, 2006.

[5] M. Ronan. *Lectures on Buildings*, volume 7 of *Perspectives in Mathematics*. Academic Press, Inc., Boston, MA, 1989.

[6] A. Schwartz and T. Shemanske. Maximal orders in central simple algebras and Bruhat-Tits buildings. *Journal of Number Theory*, 56(1):115 – 138, 1996.

[7] T. Shemanske. The arithmetic and combinatorics of buildings for Sp_n. *Transactions of the American Mathematical Society*, 359(7):3409 – 3423, 2007.

[8] J. Tits. Reductive groups over local fields. In A. Borel and W. Casselman, editors, *Automorphic Forms, Representations and L-functions. Part* 1, pages 29 – 69, Providence, RI, 1979. American Mathematical Society.